

Preliminaries to

# Versal Families of Compact Super Riemann Surfaces

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We collect some classical results on versal families of ordinary compact Riemann surfaces needed in [11].

**Grauert's theorem:**  $W$  Stein manifold,  $E, F \rightarrow W$  holomorphic vector bundles. Then  $E \simeq F$  as holomorphic vector bundles iff  $E \simeq F$  as topological vector bundles. In particular all holomorphic vector bundles on a contractible Stein manifold are trivial.

see [3] introduction.

**Cartan's theorem B:**  $\mathcal{S}$  coherent sheaf on a Stein manifold  $W$ . Then  $H^k(W, \mathcal{S}) = 0$  for all  $k \geq 1$ .

see [9] introduction.

**[8] theorem 10.5.5:** If  $\dim_{\mathbb{C}} H^i(X_y, \underline{V}_y)$  is independent of  $y \in Y$  then all sheaves  $f_{(i)}(\underline{V})$  are locally free and all maps

$$f_{y,i} : f_{(i)}(\underline{V}) / \mathfrak{m}_y f_{(i)}(\underline{V}) \rightarrow H^i(X_y, \underline{V}_y)$$

are isomorphisms.

Hereby  $f : X \rightarrow Y$  is a holomorphic family of compact complex manifolds  $X_y := f^{-1}(y)$ ,  $y \in Y$ , and  $\underline{V} \rightarrow X$  a holomorphic vector bundle.  $\underline{V}_y := \underline{V}|_{X_y}$ , and  $\mathfrak{m}_y \triangleleft \mathcal{O}_Y$  denotes the maximal ideal of all holomorphic functions vanishing at the point  $y \in Y$ . Finally  $f_{y,i} : f_{(i)}(\underline{V}) / \mathfrak{m}_y f_{(i)}(\underline{V}) \rightarrow H^i(X_y, \underline{V}_y)$  denotes the canonical homomorphism.

**Lemma 0.1**  *$W$  simply connected manifold,  $G$  group. Then  $H^1(W, G)$  is trivial.*

*Proof:* Let  $W = \bigcup_i U_i$  be an open cover and  $g_{ij} : U_i \cap U_j \rightarrow G$  locally constant,  $U_i \cap U_j \neq \emptyset$ , forming a 1-cocycle, so  $g_{ki}g_{jk}g_{ij} = 1$  if  $U_i \cap U_j \cap U_k \neq \emptyset$ , and  $g_{ii} = 1$ . We have to show that after maybe refining the open cover there exist  $h_i : U_i \rightarrow G$  locally constant such that  $g_{ij} = h_j^{-1}h_i$ .

After maybe refining the open cover we may assume by [4] corollary 5.2 that it is good, which means that all finite non-empty intersections of the  $U_i$  are diffeomorphic to  $\mathbb{R}^n$ . Then of course all  $g_{ij}$  are constant.

Choose  $i_0$  . Take an arbitrary  $i$  and a path  $\gamma : [0,1] \rightarrow W$  from  $U_{i_0}$  to  $U_i$  . Let  $I_{k\sigma}$  ,  $\sigma = 1, \dots, s_k$  , numbered increasingly, be the connected components of  $\gamma^{-1}(U_k)$  . Then  $[0,1] = \bigcup_{k,\sigma} I_{k\sigma}$  is a cover with open intervals, and there exist unique  $h_{k\sigma} \in G$  , such that  $h_{i_0,1} = 1$  and  $g_{kl} = h_{l\tau}^{-1} h_{k\sigma}$  ,  $I_{k\sigma} \cap I_{l\tau} \neq \emptyset$  . Since two such paths are homotopic and  $h_{i,s_i}$  is constant under small variation of  $\gamma$  , we see that infact  $h_i := h_{i,s_i}$  is independent  $\gamma$  .  
 $\leadsto (h_i)$  0-chain in  $G$  , and taking a path from  $U_{i_0}$  to  $U_j$  over  $U_i \cap U_j$  one sees that indeed  $g_{ij} = h_j^{-1} h_i$  .  $\square$

**Theorem 0.2**  $\pi : M \rightarrow \mathbb{R}^n$  smooth family of compact real manifolds. Then  $M$  is trivial.

*Proof:* by induction on  $n$ , inspired by [12] proof of theorem 2.4.

$n = 0$  trivial.

$n \rightarrow n+1$  : We construct a smooth vectorfield  $\chi \in H^0(TM)$  such that  $(d\pi)\chi = \partial_1 \circ \pi$  : indeed locally in  $M$  such  $\chi$  exist, so we obtain an open cover  $M = \bigcup_i U_i$  , without restriction locally finite, and  $\chi_i \in H^0(TU_i)$  with this property. Now take a smooth partition  $(\varepsilon_i)$  of unity subordinated to  $(U_i)$  and  $\chi := \sum_i \varepsilon_i \chi_i$  .

Let  $\Phi$  be the integral flow to  $\chi$  , so the maximal solution of the initial value problem  $\Phi|_{\{0\} \times M} = \text{id}_M$  and  $(d\Phi)\partial_u = \chi \circ \Phi$  . Since  $(d\pi)\chi = \partial_1 \circ \pi$  we obtain

$$\begin{array}{ccc} \mathbb{R} \times M & \xrightarrow{\Phi} & M \\ (\text{id}_{\mathbb{R}}, \pi) \downarrow & \circlearrowleft & \downarrow \pi \\ \mathbb{R} \times \mathbb{R}^{n+1} & \longrightarrow & \mathbb{R}^{n+1} \\ (u, t) & \mapsto & t + (u, 0, \dots, 0) \end{array} .$$

So since all fibers of  $M$  are compact we see that  $\Phi$  is defined on all  $\mathbb{R} \times M$  . Let  $\pi_1 : M \rightarrow \mathbb{R}$  denote the first component of  $\pi$  . Then

$$\Phi|_{\mathbb{R} \times M|_{\{0\} \times \mathbb{R}^n} : \mathbb{R} \times M|_{\{0\} \times \mathbb{R}^n} \rightarrow M$$

is a strong family morphism with inverse  $x \mapsto (\pi_1(x), \Phi(-\pi_1(x), x))$  , so a strong family isomorphism. Finally by induction hypothesis  $M|_{\{0\} \times \mathbb{R}^n}$  is trivial.  $\square$

**Theorem 0.3**  $X$  compact Riemann surface of genus  $g \geq 2$  ,  $\Phi \in \text{Aut } X$  homotopic to  $\text{id}_X$  . Then  $\Phi = \text{id}_X$  .

*Proof:* see [10].  $\square$

**Corollary 0.4**  $N \rightarrow W$  holomorphic family of compact Riemann surfaces  $Y_w$  of genus  $g \geq 2$  ,  $W$  connected,  $\Phi \in \text{Aut}_{\text{strong}} N$  ,  $w_0 \in W$  such that  $\Phi|_{Y_{w_0}} = \text{id}_{Y_{w_0}}$  . Then  $\Phi = \text{id}_N$  .

*Proof:* Since  $W$  is connected all  $\Phi|_{Y_w}$  ,  $w \in W$  , are homotopic to  $\text{id}_{Y_w}$  , so equal to  $\text{id}_{Y_w}$  by theorem 0.3.  $\square$

For every  $g \in \mathbb{N}$  let  $\mathcal{T}_g$  denote the Teichmüller space for genus  $g$  , constructed as set of equivalence classes of marked compact Riemann surfaces of genus  $g$  , see for example [1] section 2.3.

**Proposition 0.5**

- (i) For every genus  $g$  there exists a holomorphic family  $M_g \rightarrow \mathcal{T}_g$  of compact Riemann surfaces  $X_u$  such that the class of  $X_u$  equals the point  $\bar{u}$  in  $\text{Mod}_g$  .
- (ii)  $\mathcal{T}_g$  is a bounded contractible domain of holomorphy, so in particular Stein.

(iii)  $M_g$  is trivial as a smooth family of compact smooth families.

*Proof:* (i) trivial for  $g = 0$  :  $\mathcal{T}_0$  single point and  $M_0 = \widehat{\mathbb{C}}$  . Explicit construction for  $g = 1$  :  $\mathcal{T}_1$  upper half plane,  $M_1 = \mathcal{T}_1 \times \mathbb{C} / \langle S, T \rangle$  ,  $S : (u, z) \mapsto (u, z + 1)$  ,  $T : (u, z) \mapsto (u, z + u)$  . See [5] section 1 for  $g \geq 2$  .

(ii) trivial for  $g = 0$  and  $g = 1$  . For  $g \geq 2$  by [6] section 2.

(iii) trivial for  $g = 0$  and  $g = 1$  . For  $g \geq 2$  by theorem 0.2 since  $\mathcal{T}_g$  diffeomorphic to  $\mathbb{R}^{6(g-1)}$  , see [13] theorem 4.1.  $\square$

**Theorem 0.6 (Anchoring property of  $M_g$ )** *Let  $\pi_N : N \rightarrow W$  be a holomorphic family of compact Riemann surfaces  $Y_w$  of genus  $g$  ,  $W$  simply connected if  $g \geq 2$  ,  $W$  contractible and Stein if  $g \leq 1$  ,  $\sigma : Y_{w_0} \xrightarrow{\sim} X_{u_0}$  ,  $w_0 \in W$  and  $u_0 \in \mathcal{T}_g$  . Then there exists a unique  $\varphi : W \rightarrow \mathcal{T}_g$  such that  $\varphi(w_0) = u_0$  and  $\varphi$  can be extended to a fiberwise biholomorphic family morphism  $(\Phi, \varphi) : N \rightarrow M_g$  with  $\Phi|_{Y_{w_0}} = \sigma$  . If  $g \geq 2$  then also  $\Phi$  is uniquely determined by  $\sigma$  .*

*Proof:* By theorem 0.2 there exists an open cover  $W = \bigcup_i \Omega_i$  and trivializations  $\varphi_i : N|_{\Omega_i} \xrightarrow{\sim} \Omega_i \times Y_{w_0}$  as smooth families of compact smooth manifolds. On  $\Omega_i \cap \Omega_j$  ,  $\varphi_i$  and  $\varphi_j$  differ by the strong smooth family automorphism  $\varphi_j \circ \varphi_i^{-1}$  of  $(\Omega_i \cap \Omega_j) \times Y_{w_0}$  , and the homotopy class of  $(\varphi_j \circ \varphi_i^{-1})(w, \diamond) \in \text{Diff } Y_{w_0}$  is locally independent of  $w$  .  $W$  is simply connected, so  $H^1(W, (\text{Diff } Y_{w_0}) / (\text{Diff}_0 Y_{w_0})) = 0$  by lemma 0.1, where  $\text{Diff}_0 Y_{w_0}$  denotes the normal subgroup consisting of diffeomorphisms homotopic to  $\text{id}_{Y_{w_0}}$  . So after refining the open cover  $(\Omega_i)$  we may assume without restriction that all  $(\varphi_j \circ \varphi_i^{-1})(w, \diamond) \in \text{Diff}_0 Y_{w_0}$  .

Choose  $i_0$  with  $w_0 \in \Omega_{i_0}$  . Then by taking  $\left( \text{id}_{\Omega_{i_0}}, \left( \varphi_{i_0}|_{Y_{w_0}} \right)^{-1} \right) \circ \varphi_i$  instead of  $\varphi_i$  we ensure that  $\varphi_i|_{Y_{w_0}} \in \text{Diff}_0 Y_{w_0}$  whenever  $w_0 \in \Omega_i$  . From now on we identify  $N|_{\Omega_i}$  with  $\Omega_i \times Y_{w_0}$  via  $\varphi_i$  as smooth families, and for all  $w \in W$  the homotopy class of  $\varphi_i|_{Y_w} : Y_w \rightarrow Y_{w_0}$  is independent of the choice of  $i$  .

By construction of  $\mathcal{T}_g$  , for every  $w \in W$  there exists a unique  $u_w \in \mathcal{T}_g$  such that there exists an isomorphism  $Y_w \xrightarrow{\sim} X_{u_w}$  homotopic to  $\sigma$  . Of course  $u_{w_0} = u_0$  .

*Uniqueness:* Take  $(\Phi, \varphi)$  as desired. Then  $\Phi|_{Y_w} : Y_w \xrightarrow{\sim} X_{\varphi(w)}$  is homotopic to  $\sigma$  , so  $\varphi(w) = u_w$  for all  $w \in W$  . If  $g \geq 2$  then also  $\Phi$  is unique by corollary 0.4.

*Existence:*  $g \geq 2$  : By [5] section 1 there exist, locally in  $W$  , fiberwise biholomorphic family morphisms  $(\Phi, \varphi) : N \rightarrow M_g$  . Also by [5] section 1 every orientation preserving homeomorphism of  $Y_{w_0}$  induces a family automorphism of  $M_g$  , so without restriction  $\Phi|_{Y_w} : Y_w \xrightarrow{\sim} X_{\varphi(w)}$  is homotopic to  $\sigma$  , which implies  $\varphi(w) = u_w$  by construction of  $\mathcal{T}_g$  . Therefore  $\varphi$  is infact globally defined. Now on their overlaps two local choices of  $\Phi$  differ by a strong automorphism of  $\varphi^* M_g$  homotopic to  $\text{id}_{\varphi^* M_g}$  , which so equals  $\text{id}_{\varphi^* M_g}$  by theorem 0.3. We see that also  $\Phi$  is globally defined. Finally  $\Phi|_{Y_{w_0}} \circ \sigma^{-1} \in \text{Aut } X_{u_0}$  homotopic to  $\text{id}_{X_{u_0}}$  , so  $\Phi|_{Y_{w_0}} = \sigma$  by theorem 0.3.

$g = 1$  :  $(\pi_N)_* (T^{\text{rel}} N)^*$  is globally free of rank 1 by [8] theorem 10.5.5 and Grauert's theorem since  $W$  is contractible and Stein, so let  $\omega$  be a global frame such that  $\omega(w_0, \diamond) = \sigma^* dz$  . Let  $A$  and  $B$  be the pullbacks of the straightlines  $[0, 1]$  resp.  $[0, u_0]$  under  $\sigma \circ \varphi_i|_{Y_w}$  . They are closed curves in  $Y_w$  , whose homotopy classes are independent of  $i$  and generate  $\pi_1(Y_w)$  . Therefore

$$\psi, \varphi : W \rightarrow \mathbb{C} , w \mapsto \int_A \omega(w, \diamond) \text{ resp. } \int_B \omega(w, \diamond)$$

are well-defined and holomorphic on  $W$ ,  $\psi(w), \varphi(w)$  linearly independent over  $\mathbb{R}$  for all  $w \in W$ .  $\psi(w_0) = 1$  and  $\varphi(w_0) = u_0$ . After multiplying  $\omega$  with  $\frac{1}{\psi}$  we may assume that  $\psi = 1$ . Then  $\varphi(W) \subset \mathcal{T}_1$ . Locally in  $W$  take a holomorphic section  $q$  of  $N$ . So locally in  $W$ ,  $(\Phi, \varphi)$  given by

$$z \mapsto \left( \varphi(\pi_N(z)), \int_q^z \omega(\pi_N(z), \diamond) \right)$$

is a fiberwise biholomorphic family morphism  $N \rightarrow M_1$ . On their overlaps two such locally constructed  $\Phi$  differ by a translation with a holomorphic section of  $\varphi^* M_1$ . Since  $W$  contractible and Stein,  $H^1(W, \varphi^* M_1) = 0$  by Cartan's theorem B, and so  $\Phi$  can be defined globally. Finally  $\Phi|_{Y_{w_0}} \circ \sigma^{-1} \in \text{Aut } X_{u_0}$  is a translation  $\tau_a$  with some  $a \in \mathbb{C}$ , and so taking  $\tau_{-a} \circ \Phi$  instead of  $\Phi \rightsquigarrow \Phi|_{Y_{w_0}} = \sigma$ .

$g = 0$ : Locally in  $W$  take holomorphic line bundles  $L \rightarrow N$  such that  $L^{\otimes 2} = T^{\text{rel}} N$ .  $\rightsquigarrow$  open cover  $W = \bigcup_i U_i$  and holomorphic line bundles  $L_i \rightarrow N|_{U_i}$  such that  $L_i^{\otimes 2} = T^{\text{rel}} N$ . After refining the open cover we may assume that it is good, see [4] corollary 5.2.

$(L_i \otimes L_j^*)|_{Y_w}$  is trivial for all  $w \in U_i \cap U_j$ . Therefore  $(\pi_N)_*(L_i \otimes L_j^*)$  is a locally free sheaf on  $U_i \cap U_j$  of rank 1 by [8] theorem 10.5.5, so every local frame yields an isomorphism  $\psi_{ij} : L_i \xrightarrow{\sim} L_j$  locally in  $U_i \cap U_j$ , and we may assume that  $\psi_{ij}^{\otimes 2} = \text{id}_{T^{\text{rel}} N}$ . Infact we can construct a global  $\psi_{ij}$  on  $U_i \cap U_j$ : Two local choices of  $\psi_{ij}$  differ by multiplication with  $\pm 1$ , and so we see that the obstructions to define  $\psi_{ij}$  globally on  $U_i \cap U_j$  lie in  $H^1(U_i \cap U_j, \{\pm 1\})$ , which vanishes since  $U_i \cap U_j$  is contractible.

Without restriction  $\psi_{ji} = \psi_{ij}^{-1}$ . Then

$$\varphi_{ijk} := \psi_{ki} \circ \psi_{jk} \circ \psi_{ij} \in \text{Aut } L_i$$

form a 2-cocycle in  $\{\pm 1\}$ .  $H^2(W, \{\pm 1\}) = 0$  since  $W$  is contractible, and so without restriction all  $\varphi_{ijk} = 1$ . Therefore all  $L_i$  glue together to a holomorphic line bundle  $L \rightarrow N$  with  $L^{\otimes 2} = T^{\text{rel}} N$ , so in particular  $L$  is of rank 1.

By [8] theorem 10.5.5 and Grauert's theorem since  $W$  is contractible and Stein  $(\pi_N)_* L \simeq \mathcal{O}_W^{\oplus 2}$ , so let  $(f, h)$  be a global frame.  $\Phi := \frac{f}{h} \in \mathcal{M}(N)$ ,  $\Phi|_{Y_w}$  is holomorphic apart from one single pole for every  $w$ . So  $\Phi : N \rightarrow \widehat{\mathbb{C}}$  is fiberwise biholomorphic. Composing  $\Phi$  with a suitable element of  $\text{Aut } \widehat{\mathbb{C}} = PSL(2, \mathbb{C}) \rightsquigarrow \Phi|_{Y_{w_0}} = \sigma$ .  $\square$

**Theorem 0.7 (Infinitesimal universality of  $M_g$ )**  $(dM_g)_u : T_u \mathcal{T}_g \rightarrow H^1(TX_u)$  is an isomorphism for all  $u \in \mathcal{T}_g$ .

*Proof: Surjectivity:* Let  $\beta \in H^1(TX_u)$ . Since  $H^2(TX_u) = 0$ , by [12] theorem 5.6 there exists a holomorphic family  $N \rightarrow \Delta$  of compact Riemann surfaces  $Y_w$ ,  $\Delta \subset \mathbb{C}$  a disc around 0, such that  $Y_0 = X_u$  and  $\beta = (dN)_0 \partial_w$ . By the anchoring property of  $M_g$  there exists a fiberwise biholomorphic family morphism  $(\varphi, \Phi) : N \rightarrow M_g$  with  $\varphi(0) = u$ ,  $\Phi|_{Y_0} = \text{id}_{Y_0}$ , and so by the chain rule  $\beta = (dN)_0 \partial_w = (dM_g)_u (d\varphi)_0 \partial_w$ .

*Injectivity:* by equality of dimension.  $\square$

**Theorem 0.8** The modular group  $\Gamma_g$  acts properly discontinuously on  $\mathcal{T}_g$ .

*Proof:* obvious for  $g \leq 1$ , see [5] section 2 for  $g \geq 2$ .  $\square$

**Theorem 0.9**  $\{\text{id}_{M_1}, J\}$  is a set of representatives for  $(\text{Aut}_{\text{strong}} M_1) / \{\text{translations}\}$ ,

$J : (w, z) \mapsto (w, -z)$ ,

$\text{Aut}_{\text{strong}} M_2 = \{\text{id}_{M_2}, J\}$ ,  $J$  the hyperelliptic reflection, and

$\text{Aut}_{\text{strong}} M_g = \{\text{id}_{M_g}\}$  for  $g \geq 3$ .

*Proof:* simple calculation for  $M_1$  .

By [1] section 2.3 the hyperelliptic reflection  $J$  is in  $\text{Aut}_{\text{strong}} M_2 \setminus \{\text{id}_{M_2}\}$  . Let

$\Phi \in \text{Aut}_{\text{strong}} M_2$  . By [14] theorem 1 there exists  $u \in \mathcal{T}_2$  such that  $\text{Aut } X_u = \{\text{id}_{X_u}, J|_{X_u}\}$  , and so  $\Phi = \text{id}_{M_2}$  or  $\Phi = J$  by corollary 0.4.

Finally let  $\Phi \in \text{Aut}_{\text{strong}} M_g$  ,  $g \geq 3$  . By [2] there exists  $u \in \mathcal{T}_g$  such that  $\text{Aut } X_u = \{\text{id}_{X_u}\}$  , and so  $\Phi = \text{id}_{M_g}$  by corollary 0.4.  $\square$

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